Liouville theory and 2d conformal symmetry

George Jorjadze

Free University of Tbilisi

Lectures for students

Zelona Gora, December, 2019

Content

Liouville equation

Liouville theory is described by the dynamical equation

$$\partial^2_{\tau\tau}\Phi(\tau,\sigma) - \partial^2_{\sigma\sigma}\Phi(\tau,\sigma) + 4\lambda\, \mathrm{e}^{2\Phi(\tau,\sigma)} = 0$$

where τ and σ are time and space coordinates, respectively, and λ is a constant parameter.

This equation can be interpreted as a model of 2d scalar field theory with exponential self-interaction. In the light-cone coordinates $z = \tau + \sigma$, $\bar{z} = \tau - \sigma$, the Liouville equation reads

$$\partial^2_{z\bar{z}}\Phi(z,\bar{z}) + \lambda\, \mathrm{e}^{2\Phi(z,\bar{z})} = 0$$

Liouville integrated this equation in 1853 and he found its general solution

$$\Phi(\tau,\sigma) = \frac{1}{2}\,\log\frac{A'(z)\,\bar{A}'(\bar{z})}{[1+\lambda\,A(z)\,\bar{A}(\bar{z})]^2}$$

where A and $\bar{A}$ are monotonic functions ($A' > 0$, $\bar{A}' > 0$.)

Geometrical and physical interpretation

Our aim is to describe the geometrical picture related to the Liouville equation and present its physical content.

We will show that Liouville equation describes 2d space-time with constant curvature and we will interpret it as a model of 2d gravity.

We will describe the symmetries of the Liouville equation related to its geometrical picture and we will show how these symmetries define the general solution of this nonlinear field equation.

We will introduce stress tensor and asymptotic fields of Liouville theory and finally we will construct its classical S-matrix.

To describe the geometrical picture related to Liouville equation, we first introduce the curvature of 2d surfaces embedded in $\mathbb{R}^3$.

2d sarface and the quadratic forms

Let us consider a 2d surface $\vec{X}(u_1, u_2)$, where $\vec{X} \in \mathbb{R}^3$

and (u_1, u_2) are parameterizing coordinates.

The vectors $\partial_a \vec{X}$, $a = (1, 2)$, are tangent to the surface.

Their scalar products define the matrix of the first quadratic form

$$g_{ab} = \partial_a \vec{X} \cdot \partial_b \vec{X}$$

The second quadratic form is given by matrix

$$h_{ab} = \partial^2_{ab} \vec{X} \cdot \vec{N}$$

where $\vec{N}$ is the unit normal vector

$$\vec{N} = \frac{\partial_1 \vec{X} \times \partial_2 \vec{X}}{|\partial_1 \vec{X} \times \partial_2 \vec{X}|}$$

Carvature

The normal vector $\vec{N}$ is invariant under coordinate transformations

$$(u_1, u_2) \mapsto (\tilde{u}_1, \tilde{u}_2)$$

and the quadratic forms transform as follows

$$g \mapsto \tilde{g} = J^T \, g \, J \ , \qquad h \mapsto \tilde{h} = J^T \, h \, J$$

where J is the Jacobi matrix

$$J_{ab} = \frac{\partial u_b}{\partial \tilde{u}_a}$$

The scalar curvature is given by the ratio

$$R = \frac{2 \, \mathrm{det}\, [h]}{\mathrm{det}\, [g]}$$

which is invariant under coordinate transformations.

Conformal metric

The first quadratic form is also called the metric tensor.
It defines the infinitesimal length square

$$\mathsf{d}\vec{X} \cdot \mathsf{d}\vec{X} = g_{ab}\, \mathsf{d}u_a\, \mathsf{d}u_b$$

Using coordinate transformations, one can bring g_{ab} to the conformal form

$$g_{ab} = \mathsf{e}^{\phi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The corresponding coordinates are called conformal coordinates.

The vectors $\partial_1 \vec{X}$, $\partial_2 \vec{X}$ and $\vec{N}$ then form an orthogonal basis in $\mathbb{R}^3$.

Below we calculate the scalar curvature in conformal coordinates.

Calculation of the curvature

The expansion of the vectors $\partial^2_{ab}\vec{X}$ and $\partial_a\vec{N}$ in the orthogonal basis yields

$$\partial_{11}\vec{X} = \frac{\phi_1}{2}\,\partial_1\vec{X} - \frac{\phi_2}{2}\,\partial_2\vec{X} + h_{11}\,\vec{N}$$

$$\partial_{22}\vec{X} = -\frac{\phi_1}{2}\,\partial_1\vec{X} + \frac{\phi_2}{2}\,\partial_2\vec{X} + h_{22}\,\vec{N}$$

$$\partial_{12}\vec{X} = \frac{\phi_2}{2}\,\partial_1\vec{X} + \frac{\phi_1}{2}\,\partial_2\vec{X} + h_{12}\,\vec{N}$$

$$\partial_1\vec{N} = -\left(h_{11}\,\partial_1\vec{X} + h_{12}\,\partial_2\vec{X}\right)\mathsf{e}^{-\phi}$$

$$\partial_2\vec{N} = -\left(h_{21}\,\partial_1\vec{X} + h_{22}\,\partial_2\vec{X}\right)\mathsf{e}^{-\phi}$$

Calculation of the curvature

Using the identity

$$\partial_{112}\vec{X} = \partial_{121}\vec{X}$$

and comparing the coefficients of the term $\partial_2 X$ in both sides, one obtains

$$\partial_{11}\phi + \partial_{22}\phi = -2\left(\lambda_{11}\,\lambda_{22} - \lambda_{12}\lambda_{21}\right)\mathrm{e}^{-\phi}\ .$$

The scalar curvature in the conformal coordinates then reads

$$R = -\mathrm{e}^{-\phi}\ \left(\partial_{11}^2\phi + \partial_{22}^2\phi\right)$$

If the curvature is constant, $R = R_0$, the function ϕ satisfies the equation

$$\partial_{11}^2\phi + \partial_{22}^2\phi + R_0\mathrm{e}^{\phi} = 0$$

Positive constant curvature

As an example we consider a sphere of radius r embedded in $\mathbb{R}^3$

$$\vec{X} \cdot \vec{X} = r^2$$

A line-element in the coordinates of the stereographic projection is

$$\mathsf{d}\vec{X} \cdot \mathsf{d}\vec{X} = \frac{4r^4(\mathsf{d}u_1^2 + \mathsf{d}u_2^2)}{(r^2 + u_1^2 + u_2^2)^2}$$

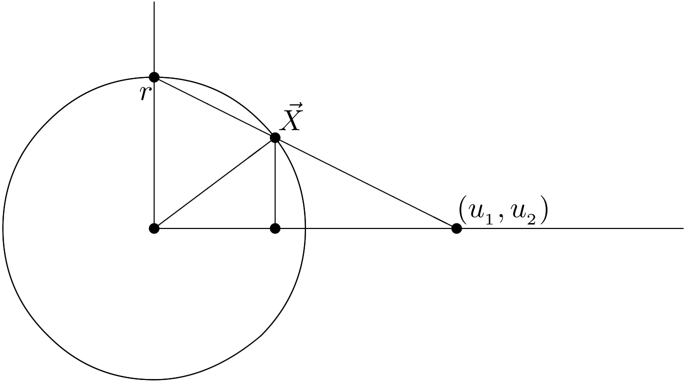
This corresponds to

$$\phi = \log \frac{4r^4}{(r^2 + u_1^2 + u_2^2)^2}$$

which leads to a positive constant curvature

$$R = R_0 = \frac{2}{r^2}$$

Stereographic projection of a sphere



$$X_1 = \frac{2r^2\,u_1}{r^2+u_1^2+u_2^2} \qquad X_2 = \frac{2r^2\,u_2}{r^2+u_1^2+u_2^2} \qquad X_3 = r - \frac{2r^3}{r^2+u_1^2+u_2^2}$$

Negative constant curvature

Let us consider the hyperbolois (Lobachevsky plane)

$$X_0^2 - X_1^2 - X_2^2 = r^2$$

A line-element here is defined by

$$\mathsf{d}l^2 = (\mathsf{d}X_1)^2 + (\mathsf{d}X_2)^2 - (\mathsf{d}X_0)^2$$

and in the coordinates of the stereographic projection one finds

$$\mathsf{d}l^2 = \frac{4r^4(\mathsf{d}u_1^2 + \mathsf{d}u_2^2)}{(r^2 - u_1^2 - u_2^2)^2}$$

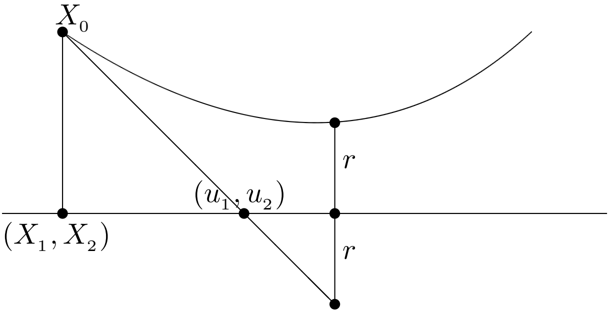
This now corresponds to

$$\phi = \log \frac{4r^4}{(r^2 - u_1^2 - u_2^2)^2}$$

and it provides a negative constant curvature

$$R = -\frac{2}{r^2}$$

Stereographic projection of a Lobachevsky plane



$X_1 = \frac{2r^2 u_1}{r^2 - u_1^2 - u_2^2}$ $\quad X_2 = \frac{2r^2 u_2}{r^2 - u_1^2 - u_2^2}$ $\quad X_0 = \frac{2r^3}{r^2 - u_1^2 - u_2^2} - r$

Here, the coordinates (u_1, u_2) are bounded on the disk

$$u_1^2 + u_2^2 < r^2$$

.

Complex conformal coordinates

Using complex conformal coordinates

$$z = u_1 + \mathrm{i}u_2 \qquad \bar{z} = u_1 - \mathrm{i}u_2$$

one obtains the line element

$$\mathrm{d}l^2 = \mathrm{e}^{\phi(z,\bar{z})}\,\mathrm{d}z\,\mathrm{d}\bar{z}$$

This form is covariant under the transformations

$$z \mapsto f(z) \qquad \bar{z} \mapsto \bar{f}(\bar{z})$$

which yields

$$\phi(z,\bar{z}) \mapsto \phi(f(z), \bar{f}(\bar{z})) + \log\left[f'(z)\bar{f}(\bar{z})\right]$$

Thus, the conformal transformations are given by the analytic functions.

Riemann tensor

The scalar curvature can also be obtained from the Riemann tensor

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ca'} \, \Gamma^{a'}{}_{bd} - \Gamma^a{}_{da'} \, \Gamma^{a'}{}_{bc}$$

where $\Gamma^a{}_{bc}$ is a symmetric connection compatible with the metric (Christoffel's symbol)

$$\Gamma^a{}_{bc} = \frac{1}{2} \, g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

A conformal metric in 2d provides the Christoffel's symbols

$$\Gamma^1{}_{11} = \Gamma^2{}_{12} = \Gamma^2{}_{21} = -\Gamma^1{}_{22} = \frac{1}{2} \, \partial_1 \phi$$

$$\Gamma^1{}_{12} = \Gamma^1{}_{21} = \Gamma^2{}_{22} = -\Gamma^2{}_{11} = \frac{1}{2} \, \partial_2 \phi$$

Scalar curvature in conformal coordinates

The non-zero components of the corresponding Riemann tensor are

$$R^1{}_{212} = -R^1{}_{221} = -R^2{}_{112} = R^2{}_{121} = -\frac{1}{2}\left(\partial_{11}^2\phi + \partial_{22}^2\phi\right)$$

The Ricci tensor $R_{bd} = R^a{}_{bad}$ then becomes

$$R_{11} = R_{22} = -\frac{1}{2}\left(\partial_{11}^2\phi + \partial_{22}^2\phi\right) \qquad R_{12} = R_{21} = 0$$

and one obtains the scalar curvature

$$R = g^{bd}\, R_{bd} = -\mathrm{e}^{-\phi}\left(\partial_{11}^2\phi + \partial_{22}^2\phi\right)$$

which coincides with the result obtained above using the quadratic forms.

Generalization to a space-time

Generalization of the Riemann geometry to Lorentzian manifolds is straightforward.

The Riemann tensor is the same, only in 2d the index $a = (0, 1)$, where $u_0 = \tau$ and $u_1 = \sigma$ correspond to time and space coordinates, respectively.

A conformal Lorentzian metric has the form

$$g_{ab} = \mathsf{e}^{\phi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the corresponding Christoffel's symbols read

$$\Gamma^0{}_{00} = \Gamma^1{}_{01} = \Gamma^1{}_{10} = \Gamma^0{}_{11} = \frac{1}{2}\,\partial_0 \phi$$

$$\Gamma^1{}_{00} = \Gamma^0{}_{01} = \Gamma^0{}_{10} = \Gamma^1{}_{11} = \frac{1}{2}\,\partial_1 \phi$$

Riemann tensor in 2d space-time

They define the Riemann tensor with non-zero components

$$R^0{}_{101} = -R^0{}_{110} = -R^1{}_{010} = R^1{}_{001} = \frac{1}{2}\left(\partial^2_{00}\phi - \partial^2_{11}\phi\right)$$

The Ricci tensor then becomes

$$R_{11} = -R_{00} = \frac{1}{2}\left(\partial^2_{00}\phi - \partial^2_{11}\phi\right) \qquad R_{01} = R_{10} = 0$$

and one obtains the scalar curvature

$$R = \mathsf{e}^{-\phi}\left(\partial^2_{00}\phi - \partial^2_{11}\phi\right)$$

Constant curvature 2d space-time

If the curvature of space-time is constant, $R = -8\lambda$, the field $\Phi = \frac{1}{2}\,\phi$ satisfies the Liouville equation

$$\partial^2_{\tau\tau}\Phi(\tau,\sigma) - \partial^2_{\sigma\sigma}\Phi(\tau,\sigma) + 4\lambda\, e^{2\Phi(\tau,\sigma)} = 0$$

Thus, the Liouville equation describes 2d space-time with constant curvature

$$R = -8\lambda$$

Positive λ corresponds to space-time with negative curvature, i.e. AdS_2.

An example for $\lambda = \mu^2$ is the (τ, σ) strip

$$\tau \in \mathbb{R}^1 \qquad 0 < \sigma < \frac{\pi}{2\mu} \qquad \mathrm{e}^{2\Phi} = \frac{1}{\sin^2(2\mu\sigma)}$$

Negative λ corresponds to a positive curvature space-time, i.e. dS_2.

It is obtained from AdS_2 by exchange of space and time coordinates.

Conformal transformations

Conformal transformations preserve the conformal form of the metric.

It is convenient to use the light-cone coordinates

$$z = \tau + \sigma \quad \bar{z} = \tau - \sigma$$

The conformal metric defines the length element

$$-\mathsf{e}^{2\Phi(z,\bar{z})}\,\mathsf{d}z\,\mathsf{d}\bar{z}$$

and the conformal transformations are given by

$$z \mapsto \zeta(z) \qquad \bar{z} \mapsto \bar{\zeta}(\bar{z})$$

with monotonic functions ζ and $\bar{\zeta}$.

The field Φ then transforms as

$$\Phi(z,\bar{z}) \mapsto \Phi(\zeta(z),\bar{\zeta}(\bar{z})) + \frac{1}{2}\,\log \zeta'(z)\,\bar{\zeta}'(\bar{z})$$

Conformal symmetry of the Liouville equation

From the geometric interpretation of the Liouville equation follows that the space of solutions is invariant under the conformal transformations, i.e. if $\Phi(z,\bar{z})$ is a solution of the Liouville equation

$$\partial^2_{z\bar{z}}\Phi(z,\bar{z}) + \lambda\,\mathrm{e}^{2\Phi(z,\bar{z})} = 0$$

then the field

$$\tilde{\Phi}(z,\bar{z}) = \Phi(\zeta(z),\bar{\zeta}(\bar{z})) + \frac{1}{2}\log\zeta'(z)\bar{\zeta}'(\bar{z})$$

satisfies the same equation.

This symmetry allows to obtain the general solution in the form presented in the introduction.

For this purpose it is enough to find one simple solution and consider its conformal transformations.

Particle in the exponential potential

A homogeneous Liouville field corresponds to $\partial_\sigma \Phi = 0$.

With the notations $Q(\tau) := \Phi(\tau, \sigma)$, $\lambda := \mu^2$ one gets the equation

$$\ddot{Q}(\tau) + 4\mu^2 \, \mathrm{e}^{2Q(\tau)} = 0$$

which corresponds to particle dynamics in the exponential potential.

Using the conservation of particle energy

$$E = \frac{1}{2} \dot{Q}^2(\tau) + 2\mu^2 \, \mathrm{e}^{2Q(\tau)}$$

after standard integration one obtains the solution

$$Q(\tau) = \log \frac{p}{2\mu \, \cosh(p\,\tau)}$$

with $p = \sqrt{2E}$.

Conformal transformations of the Liouville field

The corresponding Liouville field exponential reads

$$\mathrm{e}^{2\Phi(z,\bar z)} = \frac{p^2}{4\mu^2\,\cosh^2\left(p\,\frac{z+\bar z}{2}\right)}$$

The conforml transformations of this field can be written as

$$\mathrm{e}^{2\Phi(z,\bar z)} = \frac{A'(z)\,\bar A'(\bar z)}{[1+\mu^2\,A(z)\,\bar A(\bar z)]^2}$$

The parameterizing functions of the general solution and of the conformal transformations are related by

$$\mu\,A(z) = \mathrm{e}^{p\,\zeta(z)} \qquad \mu\,\bar A(\bar z) = \mathrm{e}^{p\,\bar\zeta(\bar z)}$$

Thus, the general solution of Liouville equation is interpreted as the orbit of the 2d conformal group.

A model of 2d gravity

The calculation of the Ricci tensor and of the scalar curvature in the conformal coordinates shows that in two dimensions Einstein equations

$$R_{ab} - \frac{1}{2}\, R\, g_{ab} = 0$$

are identities rather than equations.

On the other hand, the Liouville equation is equivalent to the covariant equation $R = R_0$ in the conformal coordinates. Note that in 2d the equation $R = R_0$ is equivalent to

$$R_{ab} - \frac{1}{2}\, R_0\, g_{ab} = 0$$

In this form the Liouville equation is interpreted as a model of 2d gravity.

Covariant action

Liouville theory can be described by the following covariant action

$$S = \int \mathsf{d}\tau \mathsf{d}\sigma \, \sqrt{-g} \, \left(\frac{1}{2} \, g^{ab} \, \partial_a \varphi \, \partial_b \varphi - 2\lambda \, \mathsf{e}^{2\varphi} - \frac{1}{2} \, \varphi \, R \right)$$

where φ is a scalar field, g_{ab} is a background metric tensor and R is its curvature. In the conformal coordinates, one has

$$g_{ab} = \mathsf{e}^{\phi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R = \mathsf{e}^{-\phi}(\phi_{\tau\tau} - \phi_{\sigma\sigma})$$

Up to the boundary and φ-independent terms, the action reduces to

$$S = \int \mathsf{d}\tau \mathsf{d}\sigma \, \left(\frac{1}{2} \, (\Phi_\tau^2 - \Phi_\sigma^2) - 2\lambda \, \mathsf{e}^{2\Phi} \right)$$

with $\Phi = \varphi + \phi/2$. Its variation obviously provides the Liouville equation.

Stress tensor

Below we consider Liouville theory for negative curvature, $\lambda = \mu^2$.

The stress tensor components of Liouville theory are introduced by

$$T = (\partial_z \Phi)^2 - \partial^2_{zz}\Phi \qquad \bar{T} = (\partial_{\bar{z}} \Phi)^2 - \partial^2_{\bar{z}\bar{z}}\Phi$$

and the energy and momentum densities can be written as

$$\mathcal{E} = T + \bar{T} = \frac{1}{2}\,(\partial_\tau \Phi)^2 + \frac{1}{2}\,(\partial_\sigma \Phi)^2 + 2\mu^2\,\mathrm{e}^{2\Phi} - \partial^2_{\sigma\sigma}\Phi$$

$$\mathcal{P} = T - \bar{T} = \partial_\tau \Phi\,\partial_\sigma \Phi - \partial^2_{\tau\sigma}\Phi$$

From the Liouville equation follows the chirality conditions

$$\partial_{\bar{z}} T = 0 \qquad \partial_z \bar{T} = 0$$

which are equivalent to the local conservation laws

$$\dot{\mathcal{E}} - \mathcal{P}' = 0 \qquad \dot{\mathcal{P}} - \mathcal{E}' = 0$$

Vertex function

The Liouville field exponential $V = e^{-\Phi}$ satisfies the equations

$$\partial^2_{zz} V = T(z)\, V \qquad \partial^2_{\bar z \bar z} V = \bar T(\bar z) V$$

which follow from the definition of the stress tensor components.
The Liouville equation is equivalent to

$$V\, \partial^2_{z\bar z} V - \partial_z V\, \partial_{\bar z} V = \mu^2$$

The solutions of the Hill equations

$$\psi''_\alpha(z) = T(z)\, \psi_\alpha(z)\ , \qquad \bar\psi''_\alpha(\bar z) = \bar T(\bar z)\, \bar\psi_\alpha(\bar z)\ , \quad (\alpha = 1, 2)\ ,$$

with unit Wronskians, $W[\psi_1, \psi_2] = W[\bar\psi_1, \bar\psi_2] = 1$, provide

$$V(x, \bar x) = \mu \left[\psi_1(x)\, \bar\psi_1(\bar x) + \psi_2(x)\, \bar\psi_2(\bar x)\right]$$

which can also be treated as the general solution in terms of chiral fields.

Asymptotic fields

Setting $T = \bar{T} = \frac{1}{4}\,p^2$, with $p > 0$, one finds again the particle solution

$$V = \mathrm{e}^{-\Phi} = \frac{\mu}{p}\left(\mathrm{e}^{-p\tau} + \mathrm{e}^{p\tau}\right)$$

and its conformal transformation can be written as

$$\mathrm{e}^{-\Phi} = \mathrm{e}^{-\Phi_{\text{in}}} + \mathrm{e}^{-\Phi_{\text{out}}} \qquad \text{with}$$

$$\Phi_{\text{in}} = \frac{1}{2}\,\log\left[\zeta'(z)\,\bar{\zeta}'(\bar{z})\right] + \frac{1}{2}\,p\left[\zeta(z) + \bar{\zeta}(\bar{z})\right] + \log\left(\frac{p}{\mu}\right)$$

$$\Phi_{\text{out}} \;= \frac{1}{2}\,\log\left[\zeta'(z)\,\bar{\zeta}'(\bar{z})\right] - \frac{1}{2}\,p\left[\zeta(z) + \bar{\zeta}(\bar{z})\right] + \log\left(\frac{p}{\mu}\right)$$

Because of $p > 0$, the fields Φ_{in} and Φ_{out} are interpreted as

the in and out fields, respectively.

The S-matrix

From the equations of the previous page follows

$$\mathsf{e}^{-\Phi_{\mathsf{out}}(z,\bar{z})} = \mathsf{e}^{-\Phi_{\mathsf{in}}(z,\bar{z})}\,\mu^2 \int_{-\infty}^{z} \mathsf{d}x\,\mathsf{e}^{2\phi_{\mathsf{in}}(x)} \int_{-\infty}^{\bar{z}} \mathsf{d}\bar{x}\,\mathsf{e}^{2\bar{\phi}_{\mathsf{in}}(\bar{x})}$$

where ϕ_{in} and $\bar{\phi}_{\mathsf{in}}$ correspond to the chiral decomposition

$$\Phi_{\mathsf{in}}(z,\bar{z}) = \phi_{\mathsf{in}}(z) + \bar{\phi}_{\mathsf{in}}(\bar{z})$$

The obtained equation defines the classical S-matrix, since it provides the out-field exponential in terms of the in field.

The aim is to quantize the system, to construct the vertex operators and to find the operator for the S-matrix.

These are the topics of additional lectures.

Conclusions

- We have introduced the notion of the scalar curvature for 2d surfaces and calculated it in the conformal coordinates.
- It has been shown that the Liouville equation describes constant curvature 2d space-time.
- We have introduced the 2d conformal transformations and it has been shown that the Liouville equation is invariant under these transformatins.
- We have shown that the general solution of the Liouville equation is given as the orbit of 2d conformal group.
- We have found that the Liouville equation can be interpreted as a model of 2d gravity.
- We have introduced the energy-momentum tensor, the vertex functions and the asymptotic fields for Liouville theory and, finally, we have constructed the classical S-matrix.

Exercises: To derive the equations presented in the lectures.